# Wave Theory II — Numerical Simulation of Waves — (3) Representation of Wave Function by Using Green Function — Eigenfunction expansion to derive the Green function in bounded region

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This lecture first treats the eigenfunction expansion method (Ohm-Rayleigh method) to derive the Green function for the ordinary differential equation in the bounded region. Next, the method is extended to the partial differential equation with the boundary conditions are separable to each of the variables.

## 1 Eigenfunction Expansion Method to Solve Inhomogenious Differential Equation: 1D Case

In this section, the solution of the inhomogenious differential equation is represented by the eigenfunctions which are the solutions of the homogenious equations, by using examples.

### 1.1 Eigenvalue and Eigenfunction

For example, the following homogenious differential equation is considered.

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi(x) + \lambda\psi(x) = 0, \tag{1}$$

$$\psi(a) = \psi(b) = 0 \ (a < b). \tag{2}$$

Equation (1) is 1D Helmholtz equation when $\lambda = k^2$. Equation (2) means that Dirichet condtion is satisfied at $x = a$ and $x = b$. These equations express the following physical phenomena:

- Current distribution of the transmission line when both ends are open,
- Voltage distribution of the transmission line when both ends are shorted.
- Latelal wave of the hard beam in free space.

This homogenious differential equation have the solution only when

$$\begin{aligned} \lambda &= \left(\frac{n\pi}{b-a}\right)^2 \\ &= \lambda_n \\ &= k_n^2, \end{aligned} \tag{3}$$

and the solution $\psi_n(x)$ is given as

$$\psi_n(x) = \sqrt{\frac{2}{b-a}} \sin k_n(x-a). \tag{4}$$

Although the coefficient is arbitrary due to homogenity, the coefficient in Eq. (4) due to the reason described in next subsection. $\lambda_n$ and $\psi_n(x)$ are called the eigenvalues and the eigenfunctions of the ordinary differential equation, respectively.

### 1.2 Complete Orthonormal Function Set

In general, set of eigenfunctions $\{\psi_n(x)\}$ has the following properties, and is called *complete orthonormal function set.*

**Orthonormal Property**

Two eigenfunctions $\psi_m(x)$ and $\psi_n(x)$ $(m \neq n)$ which have the different eigenvalues satisfy the forllowing *orthogonal relation.*

$$\int_a^b \psi_m(x)\psi_n^*(x)\mathrm{d}x = 0. \tag{5}$$

Without the loss of generality, the following *normalization condition* can be introduced.

$$\int_a^b \psi_m(x)\psi_m^*(x)\mathrm{d}x = 1. \tag{6}$$

In fact, the coefficient in Eq. (4) is defined so as to satisfy this normalization condition.

Equations (5) and (6) are combined as

$$\int_a^b \psi_m(x)\psi_n^*(x)\mathrm{d}x = \delta_{mn}, \tag{7}$$

where $\delta_{mn}$ is the Kronecker's delta. Equation (7) is called the *orthonormal condition.*

**Complete Set**

The set of the functions is called the *complete set* when arbitrary function $f(x)$ satisfying a certatin boundary condition can be expressed as

$$f(x) = \sum_n C_n \psi_n(x), \tag{8}$$

$$C_n = \int_a^b \psi_n^*(x) f(x)\mathrm{d}x. \tag{9}$$

## 1.3 Expression of Green Function by Using Eigenfunctions

Let us consider the 1D Helmholtz equation

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\phi(x) + k^2\phi(x) = -\rho(x), \tag{10}$$

which satisfies the boundary condition (2). The Green function of Eq. (10) satisfies

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}G(x,\ x') + k^2 G(x,\ x') = -\delta(x - x'). \tag{11}$$

The solution of Eq. (11) is expressed by using the set of eigenfunctions (4).

From Eq. (8), the following equation is satisfied.

$$G(x,\ x') = \sum_n C_n(x')\psi_n(x). \tag{12}$$

It is noted that $C_n$ is the function of the source point $x'$. The eiqenfunction $\psi_n(x)$ satisfies

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi_n(x) + k^2\psi_n(x) = (k^2 - k_n^2)\psi_n(x). \tag{13}$$

By substituting Eq. (12) into Eq. (11), and then Eq. (ref1Dgreen) into Eq. (13), the following relation is derived.

$$\sum_n C_n(x')(k^2 - k_n^2)\psi_n(x) = -\delta(x - x'). \tag{14}$$

Equation (14) is multiplied by $\psi_m^*(x)$ and then is integrated over the range from $x = a$ to $b$, $C_m(x')$ is derived as

$$C_m(x') = -\frac{\psi_m^*(x')}{k^2 - k_m^2}. \tag{15}$$

The final form of the Green function is given as [1]

$$G(x,\ x') = -\sum_n \frac{\psi_n(x)\psi_n^*(x')}{k^2 - k_n^2}. \tag{16}$$

Since the Green function (16) satifies the original boundary condiition (2), the socution $\phi(x)$ of the inhomogenious differntial equation (10) is given as

$$\phi(x) = \int_a^b G(x,\ x')\rho(x')\mathrm{d}x'.b \tag{17}$$

# 2 Separation of Variables and Eigenfunction: 2D Case

In this section, the eigenfunction expansion method is extended to partial differential equations. This approach is applicable only when the separation of variables is possible[2]

## 2.1 Separation of Variables

In this subsection, the ordinary differential equation (1, 2) is extended to 2D partial differential equation, as an example.

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi(x,\ y) + \lambda\psi(x,\ y) = 0, \tag{18}$$

$$\psi(a,\ y) = \psi(b,\ y) = 0\ (a < b), \tag{19}$$

$$\psi(x,\ c) = \psi(x,\ d) = 0\ (c < d), \tag{20}$$

In this case, the boundary conditions are separated into one with respect to $x$ only, and the other with respect to $y$ only. Therefore, the separation of the variables is possible as follows:

$$\psi(x,\ y) = X(x)Y(y), \tag{21}$$

$$X(a) = X(b) = 0, \tag{22}$$

$$Y(c) = Y(d) = 0, \tag{23}$$

By substituting Eq. (21) into the partial differential equation (18) is rewritten into the following two ordinary differential equations.

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}X(x) + \lambda_x X(x) = 0, \tag{24}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}y^2}Y(y) + \lambda_y Y(y) = 0, \tag{25}$$

$$\lambda_x + \lambda_y = \lambda. \tag{26}$$

For each of these equations, eigenvalues and eigenfunctions are obtained in the same manner as the 1D case as

$$\lambda_{xm} = k_{xm}^2 = \left(\frac{m\pi}{b-a}\right)^2, \tag{27}$$

$$\lambda_{yn} = k_{yn}^2 = \left(\frac{n\pi}{d-c}\right)^2, \tag{28}$$

[1] When $k^2 = k_n^2$, the differntial euqation (11) is not unique, since the homogenious solution $\psi_n$ exists. In other words, when a certain function $G_1$ satisfies Eq. (11), i.e. $G_1$ is a specific solution, $G_1 + A\psi_n$ is also a solution of Eq. (11) for arbitrary constant $A$. Therefore, the Green function $G$ is not uniquely determined.

The physical phenomenon in the case is such that the system is resonant when $k^2 = k_n^2$. When the system is lossless, i.e. $k$ is real, the magnitude of the wave function becomes infinity.

[2] In practice, however, the separation of variables is impossible to most of the realistic problems. Due to this reason, numerical simulations are important.

$$X_m(x) = \sqrt{\frac{2}{b-a}} \sin k_{xm}(x-a), \quad (29)$$

$$Y_n(y) = \sqrt{\frac{2}{d-c}} \sin k_{yn}(y-c), \quad (30)$$

As is described, *a partial differential equation is rewritten into a seet of ordinary differential equations by the separation of the variables.*

## 2.2 Derivation of Green Function

For the 2D Helmholtz equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi(x,\ y) + k^2\psi(x,\ y) = -\rho(x,\ y), \quad (31)$$

Green function is defined as

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)G(x,\ y,\ x',\ y') + k^2 G(x,\ y,\ x',\ y') = -\delta(x-x')\delta(y-y'). \quad (32)$$

In this subsection, the solution of Eq. 32 is expressed by using the set of eigenfunctions.

By using the separation of variables approach as well as the completeness of the set of eigenfunctions, the following relation is derived.

$$\begin{aligned} G(x,\ y,\ x',\ y') &= G_x(x,\ x')G_y(y,\ y') \\ &= \left(\sum_m C_{xm}(x')X_m(x)\right)\left(\sum_n C_{yn}(y')Y_n(y)\right) \\ &= \sum_m \sum_n C_{xm}(x')C_{yn}(y')X_m(x)Y_n(y). \end{aligned} \quad (33)$$

The eigenfunctions $X_m(x),\ Y_n(y)$ satisfy

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)X_m(x)Y_n(y) + k^2 X_m(x)Y_n(y) = (k^2 - k_{xm}^2 - k_{yn}^2)X_m(x)Y_n(y). \quad (34)$$

By substituting Eq. (33) into Eq. (32), and then Eq. (32) into Eq. (34), the following relation is derived.

$$\sum_m \sum_n C_{xm}(x')C_{yn}(y')(k^2 - k_{xm}^2 - k_{yn}^2)X_m(x)Y_n(y) = -\delta(x-x')\delta(y-y'). \quad (35)$$

Equation (35) is mutiplied by $X_p^*(x)Y_q^*(y)$ and then is integrated in the range of $(x,\ y)$, $C_{xp}(x')C_{yq}(y')$ is derived as

$$C_{xp}(x')C_{yq}(y') = -\frac{X_p^*(x')Y_q^*(y')}{k^2 - k_{xp}^2 - k_{yq}^2}. \quad (36)$$

The final form of the Green function is given as

$$G(x,\ y,\ x',\ y') = -\sum_m \sum_n \frac{X_m(x)Y_n(y)X_m^*(x')Y_n^*(y')}{k^2 - k_{xm}^2 - k_{yn}^2}. \quad (37)$$

Equation (37) is regarded to be in the same form as 1D case, if $\psi_{mn}(x,\ y) = X_m(x)Y_n(y)$ is the eigenfunction and $k_{mn}^2 = k_{xm}^2 + k_{yn}^2$ is the eigenvalue.

# Report

Tokyo Tech students are requested to submit by either of the following ways:

1. by passing the lecturer before the lecture, or
2. or via the mailing post of O-okayama Minami 3 bldg. 1st floor.

Do not forget to fill out the student ID, your department and lab names, as well as your name. KMITL students shall follow the instruction of Dr. Chuwong.

The handouts as well as the copies of the slides can be downloaded from the web.

`http://mobile.ss.titech.ac.jp/~takada/waves/`

## Exercises

1. Compute $G(x,\ x')$ in Eq. (16) numerically when $a=0,\ kb=12$ and source position is$x'=\frac{b}{3}$, and draw a greph. Discuss about the convergence of the infinite series with respect to $n$, by considering tha variation of $\frac{1}{k^2-k_n^2}$.

2. Present the separation of variable solution for 2D homogenious Helmholtz equation in free space in cylindrical coodinates. You can use the following relations.

   - *Laplacian in cylindrical coordinates*

   $$\nabla^2\phi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\phi}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\phi}{\partial\varphi^2} \tag{38}$$

   - *Solution of Bessel's equation*

   $$\frac{1}{\tilde{\rho}}\frac{\mathrm{d}}{\mathrm{d}\tilde{\rho}}\left(\tilde{\rho}\frac{\mathrm{d}R(\tilde{\rho})}{\mathrm{d}\tilde{\rho}}\right)+\left(1-\frac{n^2}{\tilde{\rho}^2}\right)R(\tilde{\rho})=0, \tag{39}$$

   $$R(\tilde{\rho}) = a_1H_n^{(1)}(\tilde{\rho})+a_2H_n^{(2)}(\tilde{\rho}). \tag{40}$$

3. Point out the corrections of handout, if any.