Wave Theory II — Numerical Simulation of Waves — (4) Boundary Element — (I) Boundary Integral Equation

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This lecture and the following lecture treats the boundary element method (BEM). In BEM, an integral representation of the wave function known as Kirchoff-Huygens principle is utilized to derive the bundary integral equation. Next, this integral equation is dicretized to implement in the numerical computation.

In this present lecture, the derivation of the boundary integral equations is presented by using 2D examples. The careful treatment is necessary for the singralities which occur in case that the source and the observation points are coincide.

1 Boundary Integral Equation

When the Helmholtz equation

$$\nabla^2\phi(\mathbf{r}) + k^2\phi(\mathbf{r}) = -\rho(\mathbf{r}) \tag{1}$$

and the boundary condition[1]

$$A(\mathbf{r})\hat{\mathbf{n}}\cdot\nabla\phi(\mathbf{r}) + B(\mathbf{r})\phi(\mathbf{r}) = 0, \quad \mathbf{r}\in\partial V \tag{2}$$

are given, let us consider the problem to derive the wave function $\phi(\mathbf{r})$.

Green function corresponding to Eq. (1) is defined by

$$\nabla^2 G(\mathbf{r},\ \mathbf{r}') + k^2 G(\mathbf{r},\ \mathbf{r}') = -\delta(\mathbf{r}-\mathbf{r}'). \tag{3}$$

Let us assume that $G(\mathbf{r},\ \mathbf{r}')$ satisfies the the same boundary condition as Eq. (2), i.e.,

$$A(\mathbf{r})\hat{\mathbf{n}}\cdot\nabla G(\mathbf{r},\ \mathbf{r}') + B(\mathbf{r})G(\mathbf{r},\ \mathbf{r}') = 0, \quad \mathbf{r}\in\partial V. \tag{4}$$

In case, in the integral represenation of the wave function, i.e. Kirchoff-Huygens principle

$$\phi(\mathbf{r}) = \int_V G(\mathbf{r},\mathbf{r}')\rho(\mathbf{r}')\mathrm{d}V' + \oint_{\partial V}\{G(\mathbf{r},\mathbf{r}')\nabla'\phi(\mathbf{r}') - \nabla' G(\mathbf{r},\mathbf{r}')\phi(\mathbf{r}')\}\cdot\mathrm{d}\mathbf{S}', \tag{5}$$

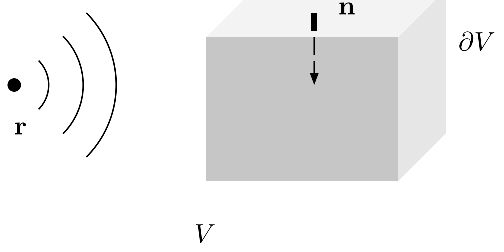

Figure 1: An environment to be considered in the derivation of the boundary integral equation

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[1] $A(\mathbf{r}) = 0$ corresponds to Dirichlet condition, $B(\mathbf{r}) = 0$ corresponds to Neumann condition, and $A(\mathbf{r})/B(\mathbf{r}) =$ const. correspoinds to impedance boundary condition.

the integrands of the sufrace integral term are identically zero. Therefore, the wavefunction is represented only by the known functions as

$$\phi(\mathbf{r}) = \int_V G(\mathbf{r}, \mathbf{r}')\rho(\mathbf{r}')\mathrm{d}V'. \tag{6}$$

However, the closed form Green function can be derived only when the variables are separated. In other words, it is impossible to derive the Green function in most of the problems. The Kirchoff-Huygens integral representation by using another Green function G_a that *does not* satisfy the boundary condition of Eq. (4) is expressed as

$$\begin{aligned}\phi(\mathbf{r}) &= \int_V G_a(\mathbf{r}, \mathbf{r}')\rho(\mathbf{r}')\mathrm{d}V' + \oint_{\partial V} \{G_a(\mathbf{r}, \mathbf{r}')\nabla'\phi(\mathbf{r}') - \nabla' G_a(\mathbf{r}, \mathbf{r}')\phi(\mathbf{r}')\} \cdot \mathrm{d}\mathbf{S}' \\ &= \int_V G_a(\mathbf{r}, \mathbf{r}')\rho(\mathbf{r}')\mathrm{d}V' + \oint_{\partial V} \left\{G_a(\mathbf{r}, \mathbf{r}')\frac{\partial\phi(\mathbf{r}')}{\partial n'} - \frac{\partial G_a(\mathbf{r}, \mathbf{r}')}{\partial n'}\phi(\mathbf{r}')\right\} \mathrm{d}S', \end{aligned} \tag{7}$$

in which the surface integral term remains.

To consider the observation point $\mathbf{r}_s$ on ∂V, the limit of $\mathbf{r}$ in Eq. (7) to $\mathbf{r}_s$ is taken as

$$\phi(\mathbf{r}_s) = \int_V G_a(\mathbf{r}_s, \mathbf{r}')\rho(\mathbf{r}')\mathrm{d}V' + \lim_{\mathbf{r}\to\mathbf{r}_s} \oint_{\partial V} \left\{G_a(\mathbf{r}, \mathbf{r}'_s)\frac{\partial\phi(\mathbf{r}'_s)}{\partial n'} - \frac{\partial G_a(\mathbf{r}, \mathbf{r}'_s)}{\partial n'}\phi(\mathbf{r}'_s)\right\} \mathrm{d}S', \tag{8}$$

where the integral variable $\mathbf{r}'_s$ is used in the surface integral term over ∂V, so as to distinguish from the volume integral variable.

The functions in Eq. (8) are categorized as

$\rho(\mathbf{r}')$	known	$\phi(\mathbf{r}_s)$	unknown
$G_a(\mathbf{r}, \mathbf{r}'_s)$	known	$\dfrac{\partial\phi(\mathbf{r}'_s)}{\partial n'}$	unknown
$\dfrac{\partial G_a(\mathbf{r}, \mathbf{r}'_s)}{\partial n'}$	known		

(8) is the integral equation with the unknown functions on the boundary $\phi(\mathbf{r}_s)$ and $\frac{\partial\phi(\mathbf{r}'_s)}{\partial n'}$, although either of the terms can be eliminated by consdering the boundary condition (2).

As Eq. (8) is *the integral equation with respect to the wave function on the surface*, it is called as *the boundary integral equation.*

2 Derivation of Boundary Integral Equation — Limiting Treatment

In this section, 2D scalar problem depicted in Fig. 2 that is uniform with respect to z-axis is considered and the boundary integral equation is derived. The scattering of acoustic or electromagnetic waves from the scatterer which is uniform and infinite along z direction can be represented.

The area S is the external region including infinity with respect to the scatter, the contour C is the boundary of the area, $\hat{\mathbf{n}}$ is the outward unit normal vector of S and is directed from C to inside the scatter.

2D free space Green function

$$G(\boldsymbol{\rho},\ \boldsymbol{\rho}') = \frac{1}{4j}\mathrm{H}_0^{(2)}(k|\boldsymbol{\rho} - \boldsymbol{\rho}'|) \tag{9}$$

is used as the Green function that does not satisfy the boundary condition.

There exists a problem of the boundary integral equation: If the observation point $\boldsymbol{\rho}$ is approaching to the boundary, Green function becomes singular for the equivalent source on the boundary when $\boldsymbol{\rho} = \boldsymbol{\rho}'$. Two alternative solutions exist to solve this problem. One is the limiting operation to derive the rigorous integral equation, which is described in the lecture. The other is the implementation technique of the numerical discretization so that the source point and the observation point are separated. This technique will be described in the following lecture after describing the descrization.

2.1 Dirichlet Condition

In this subsection, the Dirichlet boundary condition

$$\phi(\boldsymbol{\rho}) = 0, \quad \boldsymbol{\rho} \in \partial S \tag{10}$$

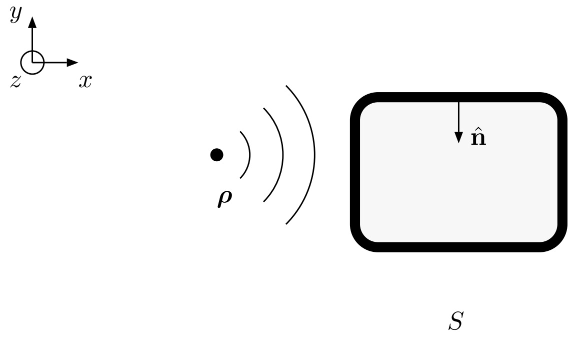

Figure 2: Numerical Model

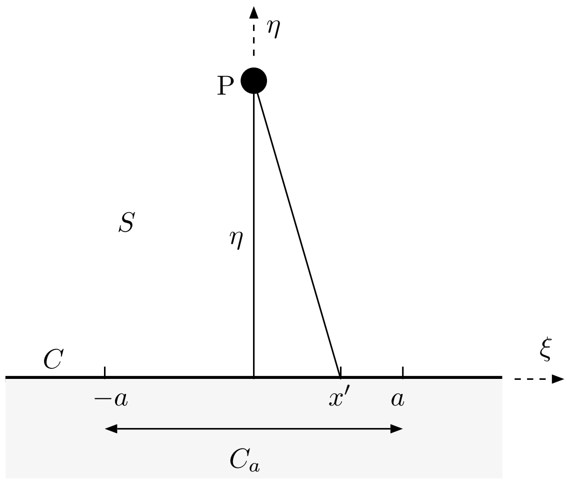

Figure 3: Local coordinates for limiting operation

is assumed[2].

It is noted that the radiation conditions

$$\lim_{|\boldsymbol{\rho}|\to\infty} \phi(\boldsymbol{\rho}) = 0 \tag{11}$$

$$\lim_{|\boldsymbol{\rho}|\to\infty} \nabla\phi(\boldsymbol{\rho}) = 0 \tag{12}$$

are satisfied so that the contour integral with respect to infinity is zero.

The integral representation is therefore expressed as

$$\phi(\boldsymbol{\rho}) = \int_S G(\boldsymbol{\rho},\ \boldsymbol{\rho}')\rho(\boldsymbol{\rho}')\mathrm{d}S' + \oint_{\partial S} G(\boldsymbol{\rho},\ \boldsymbol{\rho}')\frac{\partial\phi(\boldsymbol{\rho}')}{\partial n'}\mathrm{d}l'. \tag{13}$$

Next, the observation point $\boldsymbol{\rho}$ is assumed on the boundary so that the boundary condition Eq. (10) is represented in the inegral form.

However, the Green function becomes singular when $\boldsymbol{\rho} = \boldsymbol{\rho}'$. To avoid the singularity, the following limiting operation is applied to analytically integrate around the pole.

The integral within the infinitesimally small length C_a is considered by using the local coordinates shown in Fig. 3. As $\boldsymbol{\rho} \simeq \boldsymbol{\rho}'$, the variation of $\frac{\partial\phi}{\partial n'}$ with respect to the Green function is very small, the following approximation is valid within C_a.

$$\int_{C_a} G(\boldsymbol{\rho},\ \boldsymbol{\rho}')\frac{\partial\phi(\boldsymbol{\rho}')}{\partial n'}\mathrm{d}l' \simeq \frac{\partial\phi(\boldsymbol{\rho}')}{\partial n'}\int_{C_a} G(\boldsymbol{\rho},\ \boldsymbol{\rho}')\mathrm{d}l'. \tag{14}$$

Moreover, 2D free space Green function can be approximated as the Green function of Poisson's equation [3]

$$\nabla^2 G(\boldsymbol{\rho},\ \boldsymbol{\rho}') \simeq -\delta(\boldsymbol{\rho}-\boldsymbol{\rho}') \tag{15}$$

or

$$G(\boldsymbol{\rho},\ \boldsymbol{\rho}') \simeq -\frac{1}{2\pi}\ln|\boldsymbol{\rho}-\boldsymbol{\rho}'|, \tag{16}$$

when $\boldsymbol{\rho} \simeq \boldsymbol{\rho}'$.

Then the following integral is evaluated in the range C_a.

$$I_1 = \int_{C_a} G(\boldsymbol{\rho},\ \boldsymbol{\rho}')\mathrm{d}l'$$

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[2] This condition corresponds to the soft surface for the acoustic wave, the perfect electric conductor for E-polarized electromagnetic wave ($\phi = E_z$), or the perfect magnetic conductor for H-polarized electromagnetic wave ($\phi = H_z$).

[3] In the definition of free space Green function

$$\nabla^2 G(\boldsymbol{\rho},\ \boldsymbol{\rho}') + k^2 G(\boldsymbol{\rho},\ \boldsymbol{\rho}') = -\delta(\boldsymbol{\rho}-\boldsymbol{\rho}'),$$

$\nabla^2 G(\boldsymbol{\rho},\ \boldsymbol{\rho}') \gg k^2 G(\boldsymbol{\rho},\ \boldsymbol{\rho}')$ when $k|\boldsymbol{\rho}-\boldsymbol{\rho}'| \ll 1$.

$$
\begin{aligned}
&= -2\int_0^a \frac{1}{2\pi}\ln\sqrt{\eta^2+\xi'^2}\mathrm{d}\xi' \\
&= -\frac{1}{2\pi}\left\{a\ln\left(a^2+\eta^2\right)-2a+2\eta\tan^{-1}\frac{a}{\eta}\right\}. \qquad (17)
\end{aligned}
$$

The detail is described in Appendix A.

Moreover, by takeing the limits of $\eta \to 0$ and $a \to 0$, the following relation is obtained.

$$
\begin{aligned}
\lim_{a\to 0}\left(\lim_{\eta\to 0} I_1\right) &= \lim_{a\to 0}\left(-\frac{a}{\pi}(\ln a-1)\right), \\
&= 0. \qquad (18)
\end{aligned}
$$

By substituting the result of the limiting operation into the integral representation Eq. (13), and applying the boundary condition Eq. (10), the following boundary integral equation is obtained.

$$
0=\int_S G(\boldsymbol{\rho},\ \boldsymbol{\rho}')\rho(\boldsymbol{\rho}')\mathrm{d}S'+\int_{(\partial S-singular\ point)} G(\boldsymbol{\rho},\ \boldsymbol{\rho}')\frac{\partial\phi(\boldsymbol{\rho}')}{\partial n'}\mathrm{d}l'. \qquad (19)
$$

Equation (19) is the integral equation with the unknown function $\frac{\partial\phi(\boldsymbol{\rho}')}{\partial n'}$ which is defined on ∂S.

2.2 Newmann Condition

In this subsection, the Neumann boundary condition

$$
\hat{\mathbf{n}}\cdot\nabla\phi(\boldsymbol{\rho})=0, \quad \boldsymbol{\rho}\in\partial S \qquad (20)
$$

is assumed[4].

The integral representation is expressed as

$$
\phi(\boldsymbol{\rho})=\int_S G(\boldsymbol{\rho},\ \boldsymbol{\rho}')\rho(\boldsymbol{\rho}')\mathrm{d}S'-\oint_{\partial S}\frac{\partial G(\boldsymbol{\rho},\ \boldsymbol{\rho}')}{\partial n'}\phi(\boldsymbol{\rho}')\mathrm{d}l', \qquad (21)
$$

after applying the radiation conditions Eq. (12).

Next, the observation point $\boldsymbol{\rho}$ is assumed on the boundary so that the boundary condition Eq. (20) is represented in the integral form.

However, the normal derivative of the Green function becomes singular when $\boldsymbol{\rho}=\boldsymbol{\rho}'$. To avoid the singularity, the following limiting operation is applied to analytically integrate around the pole.

The integral within the infinitesimally small length C_a is considered by using the local coordinates shown in Fig. 3. As $\boldsymbol{\rho}\simeq\boldsymbol{\rho}'$, the variation of ϕ with respect to the normal derivative of Green function is very small, the following approximation is valid within C_a.

$$
\int_{C_a}\frac{\partial G(\boldsymbol{\rho},\ \boldsymbol{\rho}')}{\partial n'}\phi(\boldsymbol{\rho}')\mathrm{d}l'\simeq\phi(\boldsymbol{\rho}')\int_{C_a}\frac{\partial G(\boldsymbol{\rho},\ \boldsymbol{\rho}')}{\partial n'}\mathrm{d}l'. \qquad (22)
$$

Moreover, 2D free space Green function can be approximated as the Green function of Poisson's equation (16) when $\boldsymbol{\rho}\simeq\boldsymbol{\rho}'$.

By considering $n'=-\eta'$, the following integral is evaluated in the range C_a.

$$
\begin{aligned}
I_2 &= \int_{C_a}\frac{\partial G(\boldsymbol{\rho},\ \boldsymbol{\rho}')}{\partial n'}\mathrm{d}l' \\
&= 2\int_0^a\left[\frac{1}{2\pi}\frac{\partial}{\partial\eta'}\ln\sqrt{(\eta-\eta')^2+\xi'^2}\right]_{\eta'\to 0}\mathrm{d}\xi' \\
&= -\frac{1}{\pi}\tan^{-1}\frac{a}{\eta} \qquad (23)
\end{aligned}
$$

Moreover, by takeing the limits of $\eta \to 0$ and $a \to 0$, the following relation is obtained.

$$
\begin{aligned}
\lim_{a\to 0}\left(\lim_{\eta\to 0} I_2\right) &= \lim_{a\to 0}\left(-\frac{1}{2}\right) \\
&= -\frac{1}{2} \qquad (24)
\end{aligned}
$$

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[4] This condition corresponds to the hard surface for the acoustic wave, the perfect magnetic conductor for E-polarized electromagnetic wave ($\phi = E_z$), or the perfect electric conductor for H-polarized electromagnetic wave ($\phi = H_z$).

By substituting the result of the limiting operation into the integral representation Eq. (21), and applying the boundary condition Eq. (20), the following boundary integral equation is obtained.

$$\frac{1}{2}\phi(\boldsymbol{\rho}) = \int_S G(\boldsymbol{\rho},\ \boldsymbol{\rho}')\rho(\boldsymbol{\rho}')\mathrm{d}S' - \int_{(\partial S - singular\ point)} \frac{\partial G(\boldsymbol{\rho},\ \boldsymbol{\rho}')}{\partial n'}\phi(\boldsymbol{\rho}')\mathrm{d}l' \tag{25}$$

Equation (25) is the integral equation with the unknown function $\phi(\boldsymbol{\rho})$ which is defined on ∂S.

2.3 Meaning of Limiting Operation

When the observation point is outside the closed surface ∂V, the wave function given in Kirchoff-Huygens integral representation (5) is identically zero, independent of the medium. In case of Dirichlet condition, the wave function generated by the equivalent source is continuous around the source and the limiting operation do not change the integral equation. In contrast in case of Neumann condition, the wave function generated by the equivalent dipole is discontinuous and directed in opposite directions and the limiting operatation results in the average value of two discontinuous values. The latter case is similar to the limit of Fourier series at discontinuous point is the average value of both values.

Appendix A Derivation of Eq. (17)

$$I_1 = -2\int_0^a \frac{1}{2\pi}\ln\sqrt{\eta^2 + \xi'^2}\mathrm{d}\xi'$$

By converting the variables $\xi' = \eta\tan\theta$, $\mathrm{d}\xi' = \frac{\eta \mathrm{d}\theta}{\cos^2\theta}$

$$I_1 = -\frac{1}{\pi}\int_0^{\tan^{-1}\frac{a}{\eta}} \ln\sqrt{\eta^2 + \eta^2\tan^2\theta}\frac{\eta d\theta}{\cos^2\theta}.$$

Next, $1 + \tan^2\theta = \frac{1}{\cos^2\theta}$ is applied and

$$\begin{aligned} I_1 &= -\frac{1}{\pi}\int_0^{\tan^{-1}\frac{a}{\eta}} \ln\frac{\eta}{\cos\theta}\cdot\frac{\eta\mathrm{d}\theta}{\cos^2\theta} \\ &= \frac{1}{\pi}\int_0^{\tan^{-1}\frac{a}{\eta}} \ln\frac{\cos\theta}{\eta}\frac{\eta\mathrm{d}\theta}{\cos^2\theta}. \end{aligned}$$

Integrating by parts,

$$\begin{aligned} I_1 &= \frac{1}{\pi}\left\{\left[\ln\frac{\cos\theta}{\eta}\cdot\eta\tan\theta\right]_0^{\tan^{-1}\frac{a}{\eta}} + \int_0^{\tan^{-1}\frac{a}{\eta}}\eta\tan^2\theta\mathrm{d}\theta\right\} \\ &= \frac{1}{\pi}\left\{a\ln\frac{\cos\left(\tan^{-1}\frac{a}{\eta}\right)}{\eta} + \eta\int_0^{\tan^{-1}\frac{a}{\eta}}\left(\frac{1}{\cos^2\theta} - 1\right)\mathrm{d}\theta\right\} \\ &= \frac{1}{\pi}\left\{-\frac{a}{2}\ln\left(\frac{\eta^2}{\cos^2\left(\tan^{-1}\frac{a}{\eta}\right)}\right) + \eta\left[\tan\theta - \theta\right]_0^{\tan^{-1}\frac{a}{\eta}}\right\} \\ &= \frac{1}{\pi}\left\{-\frac{a}{2}\ln\left(\eta^2\left(1 + \tan^2\left(\tan^{-1}\frac{a}{\eta}\right)\right)\right) + a - \eta\arctan\frac{a}{\eta}\right\} \\ &= -\frac{1}{2\pi}\left\{a\ln\left(a^2 + \eta^2\right) - 2a + 2\eta\tan^{-1}\frac{a}{\eta}\right\}. \end{aligned}$$

Report

Tokyo Tech students are requested to submit by either of the following ways:

1. by passing the lecturer before the lecture, or

2. or via the mailing post of O-okayama Minami 3 bldg. 1st floor.

Do not forget to fill out the student ID, your department and lab names, as well as your name. KMITL students shall follow the instruction of Dr. Chuwong.

The handouts as well as the copies of the slides can be downloaded from the web.

`http://mobile.ss.titech.ac.jp/~takada/waves/`

Exercises

1. Derive Eq. (23) by referring Appendix A.
2. Prove the following relation.

$$\int_V G(\mathbf{r},\mathbf{r}')\rho(\mathbf{r}')\mathrm{d}V' + \oint_{\partial V} \{G(\mathbf{r},\mathbf{r}')\nabla'\phi(\mathbf{r}') - \nabla' G(\mathbf{r},\mathbf{r}')\phi(\mathbf{r}')\} \cdot \mathrm{d}\mathbf{S}' = 0, \quad \mathbf{r} \notin V. \tag{26}$$

References

[1] N. Morita, N. Kumagai and J.R. Mautz: **Integral Equation Methods for Electromagnetics**, Artech House (1990)

[2] R. F. Harrington: **Field Computation by Moment Methods**, Macmillan (1968) / IEEE Press (1993)