Wave Theory II — Numerical Simulation of Waves — (5) Boundary Element Method — (II) Discretization of Integral Equation and Examples

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In this lecture, the numerical solution of the boundary integral equation is presented. The boundary integral equation is discretized into an linear algebraic equations. The approach in this lecture follows the scheme known as the method of moment (MoM) which is popular in electromagnetic waves, although it is almost identical to the boundary element method (BEM). Then the application example of the discretization is described, although it is not a BEM example.

1 Discretization of Integral Equation

1.1 Boundary Integral Equation

After applying limiting operation, the boundary integration equations for 2D case with Dirichlet condition and Newmann condition can be presented as

$$\oint_{(\partial S-\text{singular point})} G(\rho,\rho')\frac{\partial\phi(\rho')}{\partial n'}\mathrm{d}l' = -\int_S G(\rho,\rho')\rho(\rho')\mathrm{d}S', \tag{1}$$

$$\frac{1}{2}\phi(\rho) + \oint_{(\partial S-\text{singular point})} \frac{\partial G(\rho,\rho')}{\partial n'}\phi(\rho')\mathrm{d}l' = \int_S G(\rho,\rho')\rho(\rho')\mathrm{d}S'. \tag{2}$$

In Eqs. (1) and (2), variables $\frac{\partial\phi(\rho)}{\partial n}$ and $\phi(\rho)$ are unknown functions on the boundary, respectively.

1.2 Method of Moments

For simplicity, integral iquation ais written as a functional of unknown function f as

$$Lf = g, \tag{3}$$

where L is a linear operator to f [1] including integrals, and g is a known function. Relations with Eqs. (1) and (2) are listed in Table 1.

In BEM/MoM, discretization of the boundary integral equation can be divided into two steps, as follows:

1. discretization of the unknown function,
2. discretization of the equation itself.

Table 1: eq. Relations between (1)/ (2) and (3)

(3)	(1)	(2)
f	$\frac{\partial\phi(\boldsymbol{\rho}')}{\partial n'}$	$\phi(\boldsymbol{\rho}')$
L	$\int_{(\partial S-\text{singular point})} G(\boldsymbol{\rho},\ \boldsymbol{\rho}')\cdot\mathrm{d}l'$	$\int_{(\partial S-\text{singular point})} \frac{\partial G(\boldsymbol{\rho},\ \boldsymbol{\rho}')}{\partial n'}\cdot\mathrm{d}l' + \frac{1}{2}\int_{\text{singular point}} \delta(\boldsymbol{\rho}-\boldsymbol{\rho}')\cdot\mathrm{d}l'$
g	$-\phi^{\text{inc}}(\boldsymbol{\rho})$	$\phi^{\text{inc}}(\boldsymbol{\rho})$

[1] $L(f_1+f_2) = Lf_1 + Lf_2,\ L(\alpha f) = \alpha L(f)$

Discretization of unknown function First, unknown function f can be approximated by using combination of the appropriate number N of known function f_n $(n = 1, 2, \ldots, N)$ as

$$f \simeq \sum_{n=1}^{N} \alpha_n f_n, \tag{4}$$

where the function f_n is called a *basis function* or an *expansion function*, and α_n is an unknown coefficient.

If we substitute Eq. (4) into (3), we can get the following approximation.

$$\sum_{n=1}^{N} \alpha_n L f_n \simeq g. \tag{5}$$

In Eq. (5), the problem to obtain an unknown function f is reduced to the problem to obtain a set of unknown coefficients $\{\alpha_n\}$.

Discretization of the equation itself Equation (5) itself is still satisfied continuously in the domain D or ∂V and is not yet an algebraic equation.

By taking the inner product of of Eq. (5) with the appropriate number $M(\geq N)$ known functions w_m $(m = 1, 2, \ldots, M)$, we can get

$$\sum_{n=1}^{N} \alpha_n < w_m, L f_n >=< w_m, g >, \quad m = 1, 2, \ldots, M, \tag{6}$$

where w_m is called a *testing function* or a *weighting function*, and $<,>$ is an inner product between two function or a moment defined by

$$< \phi, \psi >= \int_D \phi^*(\mathbf{r}) \psi(\mathbf{r}) \mathrm{d}\mathbf{r}. \tag{7}$$

In Eq. (6), all the moments are already known, and therefore Eq. (6) compries a set of linear algebraic equations.

In matrix form, it can be shown as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ a_{M1} & a_{M2} & \cdots & a_{MN} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_N \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ \vdots \\ b_M \end{bmatrix}, \tag{8}$$

$$a_{mn} = < w_m, L f_n >, \tag{9}$$

$$b_m = < w_m, g > . \tag{10}$$

Eq. (8) can be solved by using the matrix inversion when $M = N$, while a more robust result can be obtained by using least-squares method when $M > N$.

1.3 Basis Function and Testing Function

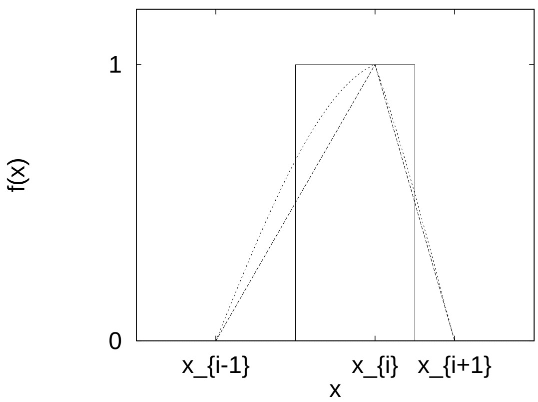
In MoM, there are many choices of the basis and the testing functions.

Basis and testing functions can be defined both in entire domain or only in sub-domain[2]. Traditionally, robust basis functions have been used with the sacrifice of the complicated mathematical maanipulations.

For basis function in sub-domain, δ function, step function, pulse function, triangular function, quadratic function, or piecewise sinusoidal function are used. For basis function in entire domain, eigen-functions such as orthogonal polynomials e.g. Tchebycheff polynomials, or trigonometric function are used. In general, if shape of basis function is similar to the actual solution, only a few number of basis functions are necessary. In Fig. 1, an example of sub-domain basis function is presented.

For testing function, δ function or basis function is used, which is called the point matching method or the Galerkin's method, respectively. If appropriate basis functions are chosen, Galerkin's method will give the better accuracy than the point mathing method with small number of unknowns, although the computation for the former case is more complicated than the latter.

[2]In BEM, only sub-domain basis is used due to the definition of *element*.



1. step function

$$f_n(x) = \begin{cases} 1, & \frac{1}{2}(x_{i-1} + x_i) \le x \le \frac{1}{2}(x_i + x_{i+1}) \\ 0, & \text{otherwise} \end{cases}$$

2. triangle function

$$f_n(x) = \begin{cases} 1 - \frac{x_n - x}{x_n - x_{n-1}}, & x_{n-1} \le x \le x_n \\ 1 - \frac{x - x_n}{x_{n+1} - x_n}, & x_n \le x \le x_{n+1} \\ 0, & \text{otherwise} \end{cases}$$

3. piecewise sinusoidal function

$$f_n(x) = \begin{cases} \frac{\sin k(x - x_{n-1})}{\sin k(x_n - x_{n-1})}, & x_{n-1} \le x \le x_n \\ \frac{\sin k(x_{n+1} - x)}{\sin k(x_{n+1} - x_n)}, & x_n \le x \le x_{n+1} \\ 0, & \text{otherwise} \end{cases}$$

Figure 1: Example of basis function

2 Numerical Example

The following is not the example of BEM but FEM. However, this is a helpful example to understand the discretization process. In the example, 1D wave is considered, and the problem is to find the excitation condition.

> Solve the following problem with respect to $f(x)$ in the domain of $0 < x < 1$ by using MoM. Use the pulse function as a basis function and δ function as a testing function.
>
> $$\begin{aligned} \int_0^1 g(x, x') f(x') \mathrm{d}x' &= \sin \pi x \\ g(x, x') &= \begin{cases} \sin x \sin(1 - x'), & x < x' \\ \sin(1 - x) \sin x', & x > x' \end{cases} \end{aligned}$$
>
> It is noted that the rigorous solution of this integral equation is $f(x) = (\text{constant}) \sin \pi x$.

In fact, g is 1D Green function that satisfies the Dirichlet condition at $x =$ and $x = 1$.

2.1 Formularization

2.1.1 Basis Function and Testing Function

In case of N-segment basis function $f_n(x)$ and testing function $w_m(x)$, both can be define as

$$\Delta x = \frac{1}{N}, \tag{11}$$

$$x'_n = n\Delta x, \tag{12}$$

$$x_m = \left(m - \frac{1}{2}\right) \Delta x, \tag{13}$$

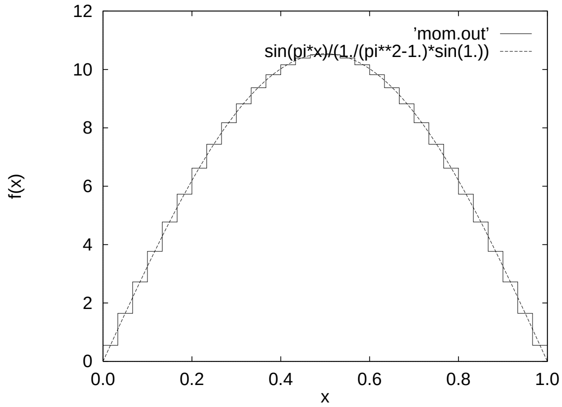


Figure 2: Numerical result

$$f_n(x) = \begin{cases} 1, & x'_{n-1} < x < x'_n, \\ 0, & \text{otherwise}, \end{cases} \quad (14)$$

$$w_m(x) = \delta(x - x_m). \quad (15)$$

It is noted that the basis function is the pulse function, and the testing function is the delta function.

2.2 Calculation of Moment

In MoM, the equation can be discretized in the following manner.

$$\sum_{n=1}^{N} \alpha_n \int_0^1 \int_0^1 w_m(x) g(x, x') f_n(x') \mathrm{d}x \mathrm{d}x' = \int_0^1 w_m(x) \sin \pi x \mathrm{d}x. \quad (16)$$

Accordingly, the following values shall be obtained.

$$a_{mn} = \int_0^1 \int_0^1 w_m(x) g(x, x') f_n(x') \mathrm{d}x \mathrm{d}x', \quad (17)$$

$$b_m = \int_0^1 w_m(x) \sin \pi x \mathrm{d}x. \quad (18)$$

In Eqs. (17) and (18), analytical is possible and

$$a_{mn} = \begin{cases} \sin x_m \{\cos(1 - x'_n) - \cos(1 - x'_{n-1})\}, & x_m < x'_{n-1}, \\ \sin x_m \{\cos(1 - x_m) - \cos(1 - x'_{n-1})\} + \sin(1 - x_m)\{-\cos x'_n + \cos x_m\}, & x'_{n-1} < x_m < x'_n, \\ \sin(1 - x_m)\{-\cos x'_n + \cos x'_{n-1}\}, & x_m > x'_n, \end{cases} \quad (19)$$

$$b_m = \sin \pi x_m. \quad (20)$$

In general, these moments are very hard to be calculated analytically, and the numerical integration is used. In the case, the number of segments in the numerical integration shall be inversely proportional to the distance between the source and the observation point. It is also the case that source and observation points are intentionally seprated to avoid the limiting operation.

2.2.1 Numerical Result

The numerical result with the segment number of 30 is shown in Fig. 2, together with the regorous solution. Within the range of x, numerical result gives the good prediction.

3 Practical Applications

The difference between BEM and some other methods like finite element method (FEM) is that unknown function is located at the boundary in BEM. If we put a homogeneous scatterer in a homogeneous medium, then the use of BEM is more profitable than FEM. Contrary, if the medium is not homogeneous or is changing gradually, we can not define the boundary integral equation and therefore we can not use BEM. Furthermore, when the object is large enough compared with the wavelength, there is some possibility that the integral equation becomes singular, i.e. the resonance. To avoid this problem, the combination of two integral equations is proposed, which is known as the combined field integral equation (CFIE).

In the field of electromagnetic waves , BEM/MoM is widely used for the simulations of linear antennas, waveguides, printed circuit boards, and schattering objects. Moreover, eigenanalysis is applied to determine the wavenumber of the waveguiding structures, and the Q factor of the resonators.

Report

Do not forget to fill out the student ID, your department and lab names, as well as your name.

The handouts as well as the copies of the slides can be downloaded from the web.

`http://mobile.ss.titech.ac.jp/~takada/waves/`

Exercises

1. Impliment the computer code to solve the scattering problem described as below:
   2D problem is assumed. A hard circle with radius a is located at the origin of xy-plane in the free space. Time-harmonic plane wave propagating to $+x$ direction is incident. By using BEM/MoM, compute $\dfrac{\partial \phi}{\partial n}$ on the surface. Variables and their values shall by defined by yourselves.

References

[1] N. Morita, N. Kumagai and J. R. Mautz: **Integral Equation Methods for Electromagnetics**, Artech House (1990)

[2] R. F. Harrington: **Field Computation by Moment Methods**, Macmillan (1968) / IEEE Press (1993)