Wave Theory II
(6) Finite Element Method

Jun-ichi Takada (takada@ide.titech.ac.jp)

In this lecture, the finite element method (FEM) is described. The Helmholtz equation and the boundary condition are transformed into a new volume integral equation by using the method of weighted residual. This approach is very similar to the method of moment which has been used in the boundary element method. Next, the volume integral equation is discretized by using FEM. Finally, the element which is a significant feature of FEM is described.

1 Method of Weighted Residual

The method of weighted residual is a method to derive the volume integral equation from the partial differential equation and the boundary condition.

Figure 1 shows the problem under consideration. The wave function $\phi(\mathbf{r})$ satisfies the Helmholtz equation

$$\nabla^2\phi(\mathbf{r}) + k^2\phi(\mathbf{r}) = -\rho(\mathbf{r}) \tag{1}$$

as well as the boundary condition

$$\phi(\mathbf{r}) = 0, \qquad \mathbf{r} \in \partial V_D, \tag{2}$$

$$\frac{\partial\phi(\mathbf{r})}{\partial n} = 0, \qquad \mathbf{r} \in \partial V_N = \partial V - \partial V_D, \tag{3}$$

where n is the outward normal direction of ∂V from the domain V. Equations (2) and (3) are the Dirichlet and the Neumann boundary conditions, respectively. ∂V_D and ∂V_N are the portions of the boundary that satisfy the Dirichlet and the Neumann boundary conditions.

Let us consider a function $\psi(\mathbf{r})$ that satisfies the Dirichlet condition (2) in order to approximate $\phi(\mathbf{r})$. The residuals of the approximation for Helmholtz equation (1) and Neumann condition (3) are defined as

$$R_V(\mathbf{r}) = \nabla^2\psi(\mathbf{r}) + k^2\psi(\mathbf{r}) + \rho(\mathbf{r}), \tag{4}$$

$$R_{\partial V_n}(\mathbf{r}) = -\frac{\partial\psi(\mathbf{r})}{\partial n}. \tag{5}$$

In order that the residuals (4) and (5) are zero in V and ∂V_N respectively in average, the following weighted averages of the residuals are considered. Equations (4) and (5) are averaged by using the weighting function $w(\mathbf{r})$ which satisfies the Dirichlet condition (2), and the sum of them are assumed to be zero as

$$\int_V w(\mathbf{r})R_V(\mathbf{r})\mathrm{d}V + \oint_{\partial V} w(\mathbf{r})R_{\partial V_n}(\mathbf{r})\mathrm{d}S = 0. \tag{6}$$

By using the Green's first theorem

$$\int_V (f\nabla^2 g + \nabla f \cdot \nabla g)\mathrm{d}V = \oint_{\partial V} f\frac{\partial g}{\partial n}\mathrm{d}S, \tag{7}$$

Eq. (6) can be modified into

$$\int_V \left(\nabla w(\mathbf{r}) \cdot \nabla\psi(\mathbf{r}) + k^2 w(\mathbf{r})\psi(\mathbf{r})\right)\mathrm{d}V = -\int_V w(\mathbf{r})\rho(\mathbf{r})\mathrm{d}V. \tag{8}$$

Equation (8) is regarded as an integral equation with a known function $w(\mathbf{r})$ and an unknown function $\psi(\mathbf{r})$.

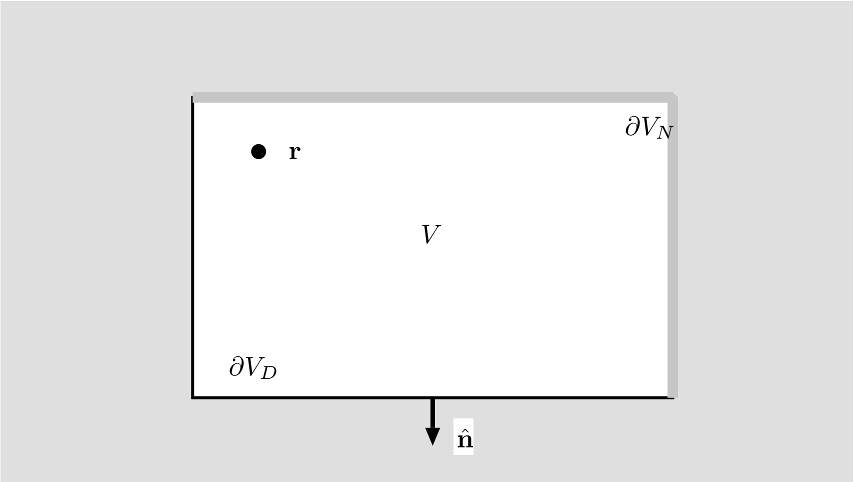

Figure 1: Problem to be considered for the volume integral equation

2 Comparison with the Approach from Variational Method

In general, the finite element method is known as the discrete version of the variational method to minimize the energy functional of the total system instead of solving the Helmholtz equation.

However, the functional does not exist if there exists the loss of the energy, i.e. the problem including the lossy medium. Contrary, the approach in the previous section used the method of weighted residual to the partial differential equation. The discrete version of this method is also called the finite element method.

When the energy functional exists, it is proved that the latter coincides with the former.

Although most of the textbooks on FEM only describe the former approach, the latter approach is chosen in the lecture to consider the versatility and the consistency with BEM.

The importance of the term FEM is not the derivation process of the equations but the element which is locally defined a function or a function with a compact support. In the sense, BEM is sometimes classified as a version of FEM.

3 Finite Element Method (FEM)

In FEM, the whole domain of the integral equation (8) is divided into a lot of polyhedra or polygons subdomains which are called *elements*. Next, the unknown function ψ is approximated by the sum of the basis functions which have the nonzero value only within the corresponding elements. Next, the Gelerkin's method is applied for the weighting function to discretized the integral equation into the linear equation. As far s the size of the element is sufficiently small, the basis function can be as simple as linear or quadrature functions.

3.1 1D Case

For 1D problem, the integral equation (8) is expressed as

$$\int_a^b \left(w'(x)\psi'(x) + k^2 w(x)\psi(x)\right) \mathrm{d}x = -\int_a^b w(x)\rho(x)\mathrm{d}x. \tag{9}$$

Either the Dirichlet or the Neumann boundary condition is satisfied at $x = a$ and $x = b$.

The domain $a \le x \le b$ is then divided into the elements. For 1D problem, each of the element is a linear segment. N evenly divided segments are considered within the range of $a \le x \le b$. Hare, x_n and ϕ_n are defined as

$$x_0 = a, \tag{10}$$
$$x_N = b, \tag{11}$$

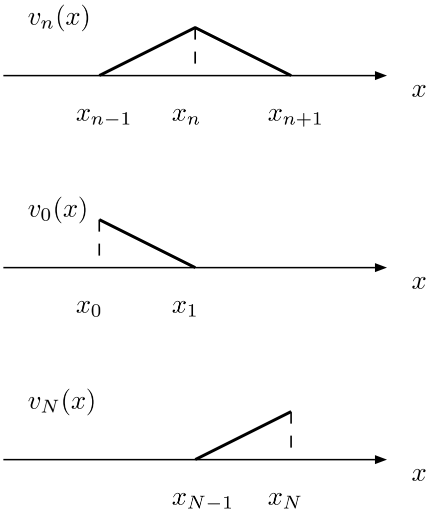

Figure 2: 1D Basis function in FEM

$$
\begin{aligned}
\Delta x &= \frac{b-a}{N}, && (12)\\
x_n &= x_0 + n\Delta x, && (13)\\
\phi_n &= \psi(x_n). && (14)
\end{aligned}
$$

The basis and weighting function of the Galerkin's method is defined with respect to each of the segment so that the value at the point under consideration is 1 and the values at both neighboring sides are 0. If the function is piecewise linear within the element, $v_n(x)$ is defined as[1].

- Internal domain $(0 < n < N)$

$$
v_n(x) = \begin{cases} \frac{x-x_{n-1}}{\Delta x}, & x_{n-1} \le x \le x_n \\ \frac{x_{n+1}-x}{\Delta x}, & x_n \le x \le x_{n+1} \\ 0, & x \le x_{n-1} \text{ or } x \ge x_{n+1} \end{cases}, \qquad (15)
$$

- Boundary ($n = 0$ and $n = N$)

$$
v_0(x) = \begin{cases} \frac{x_1-x}{\Delta x}, & x_0 \le x \le x_1 \\ 0, & x \ge x_1 \end{cases}, \qquad (16)
$$

$$
v_N(x) = \begin{cases} \frac{x-x_{N-1}}{\Delta x}, & x_{N-1} \le x \le x_N \\ 0, & x \le x_{N-1} \end{cases}. \qquad (17)
$$

Figure 2 shows $v_n(x)$.

Here, $v'_n(x)$, first derivative of $v_n(x)$, is expressed as [2]

- Internal domain $(0 < n < N)$

$$
v'_n(x) = \begin{cases} \frac{1}{\Delta x}, & x_{n-1} < x < x_n \\ -\frac{1}{\Delta x}, & x_n < x < x_{n+1} \\ 0, & x < x_{n-1} \text{ or } x > x_{n+1} \end{cases} \qquad (18)
$$

- Boundary ($n = 0$ and $n = N$)

$$
v'_0(x) = \begin{cases} -\frac{1}{\Delta x}, & x_0 < x < x_1 \\ 0, & x > x_1 \end{cases} \qquad (19)
$$

$$
v_N`(x) = \begin{cases} \frac{1}{\Delta x}, & x_{N-1} < x < x_N \\ 0, & x < x_{N-1} \end{cases} \qquad (20)
$$

---

[1] When the basis function is given as a linear function, the element is called linear element. In the practical wave scattering problems, quadratic elements are usually used.

[2] In practice, $v'_n(x)$ is infinity at $x = x_{n-1},\ x = x_n,\ x = x_{n+1}$. However, the approximated $\psi(x)$ does not have such a singularity. Therefore, the effects of these singularities can be negligible.

Next, $\psi(x)$ and $\psi'(x)$ are approximated by using $v_n(x)$ and $v'_n(x)$ as

$$\begin{aligned} \psi(x) &= \sum_{n=0}^{N} \phi_n v_n(x), & (21) \\ \psi'(x) &= \sum_{n=0}^{N} \phi_n v'_n(x). & (22) \end{aligned}$$

By substituting these values into the integral equation (8), and applying the Gelerkin's method $w(x) = v_m(x)$ $(1 \leq m \leq N-1)$ the following linear equation is derived.

$$\int_{x_0}^{x_N} \left\{ v'_m(x) \sum_{n=0}^{N} \phi_n v'_n(x) + k^2 v_m(x) \sum_{n=0}^{N} \phi_n v_n(x) \right\} \mathrm{d}x = -\int_{x_0}^{x_N} v_m(x)\rho(x)\mathrm{d}x. \quad (23)$$

The integral term of Eq. (23) is expressed as

$$\begin{aligned} K_{mn} &= \int_{x_0}^{x_N} v'_m(x) v'_n(x) \mathrm{d}x = \left\{ \begin{array}{cl} \frac{2}{\Delta x}, & n = m \\ -\frac{1}{\Delta x}, & |n-m| = 1 \\ 0, & |n-m| \geq 2 \end{array} \right. . & (24) \\ M_{mn} &= \int_{x_0}^{x_N} v_m(x) v_n(x) \mathrm{d}x = \left\{ \begin{array}{cl} \frac{2}{3}\Delta x, & n = m \\ \frac{1}{6}\Delta x, & |n-m| = 1 \\ 0, & |n-m| \geq 2 \end{array} \right. , & (25) \\ q_m &= \int_{x_0}^{x_N} v_m(x)\rho(x)\mathrm{d}x. & (26) \end{aligned}$$

Therefore, Eq. (23) is expressed in the matrix form as

$$(\mathbf{K} + k^2\mathbf{M})\boldsymbol{\phi} = \mathbf{q}. \quad (27)$$

It is noted that the coefficient matrix in Eq. (27) is $(N-1) \times (N+1)$ and no unique solution exists, since the boundary conditions at $n = 0$ and $n = N$ are not yet considered. For example, the boundary conditions of ϕ_0 can be expressed in the following way.

Dirichlet condition Simply $\phi_0 = 0$.

Neumann condition In case, $\psi'(0) \simeq \phi_0 v'_0(0) + \phi_1 v'_1(0) = 0$ is satisfied from Eq. (22), and therefore $\phi_0 = \phi_1$.

These kinds of conditions are substituted into ϕ_0 and ϕ_N. Then, Eq. (27) is modified so that the coefficient matrix is $(N-1) \times (N-1)$. Therefore, the equation can be solved with respect to ϕ.

3.2 2D Case

The same approach can be used as 1D, but the definition of the adjacent nodes is different. In 2D case, the domain is the area, and the neighboring 3 points compose a triangle as an element. Each of the basis and weighting function is defined with respect to a point and all the elements including the point as the node. The basis and weighting function with respect to a single element is defined so that the value at the node under consideration is 1 and the values at the other two nodes are 0. For the linear element, the basis and weighting function within the element is linear (plane). Figure 3 shows the 2D element, as well as the basis and weighting function.

The linear function within an element is derived as the following manner. An arbitrary linear function can be expressed as

$$v(x, y) = a + bx + cy. \quad (28)$$

The position of the node under consideration is set as (x_0, y_0), and those for the other nodes are (x_1, y_1) and (x_2, y_2). In case, a, b, c shall satisfy the following equation.

$$\begin{bmatrix} 1 & x_0 & y_0 \\ 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \quad (29)$$

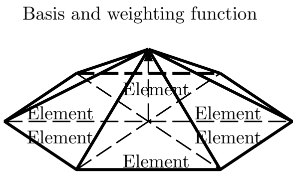

Figure 3: 2D element and its basis and weighting function.

Therefore, the equation of the plane (28) is expressed in the matrix form as

$$v(x,y) = \begin{bmatrix} 1 & x & y \end{bmatrix} \begin{bmatrix} 1 & x_0 & y_0 \\ 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \tag{30}$$

In the same manner, the components of $\nabla v(x,y)$ can be expressed as

$$\frac{\partial v(x,y)}{\partial x} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & x_0 & y_0 \\ 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \tag{31}$$

$$\frac{\partial v(x,y)}{\partial y} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & x_0 & y_0 \\ 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \tag{32}$$

Although the following process is not described, the same procedure is taken to get the set of linear equations.

3.3 Automatic Mesh Generation

One of the biggest advantage of FEM is its flexibility in mesh generation. As far as triangular elements are used, an efficient mesh generation scheme such as Delaunay triangulation. Those who have the interests in Delaunay triangulation are suggested to refer the textbooks of computational geometry [3].

Report

Do not forget to fill out the student ID, your department and lab names, as well as your name.

The handouts as well as the copies of the slides can be downloaded from the web.

`http://mobile.ss.titech.ac.jp/~takada/waves/`

Exercises

1. Solve the following 1D Helmholtz equation defined within $0 \leq x \leq \frac{12}{k}$

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\phi(x) + k^2\phi(x) = -\rho(x)$$

with the boundary condition

$$\phi(0) = \phi\left(\frac{12}{k}\right) = 0$$

and the excitation condition

$$\rho(x) = \delta\left(x - \frac{3}{k}\right)$$

by using FEM. Note that this problem is identical to the exercise in handout No. (3).

References

[1] E. Yamashita (Eds), **Analysis Methods for Electromagnetic Wave Problems**, Chap. 1, Artech House (1990).

[2] J. Jin, **The Finite Element Method in Electromagnetics**, 2nd Eds., John Wiley & Sons (2002).

[3] For example, F. P. Preparata and M. I. Shamos, **Computational Geometry: An Introduction**, Springer Verlag (1991).