Wave Theory II
(8) Finite Difference Method

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In this lecture, the finite difference method (FDM) is described. The partial differential equation (i.e. Helmholtz equation) is discretized at the grid point by using the finite difference approximation. Therefore, FDM is classified as a domain method. The classification of the differential equations is first described, and then the discretization for the elliptic partial differential equation (i.e. Helmholtz equation) is described.

1 Classification of Differential Equations

The second order partial differential equation (PDE) with respect to x and y is expressed in the general form as

$$a\frac{\partial^2\phi}{\partial x^2} + b\frac{\partial^2\phi}{\partial x\partial y} + c\frac{\partial^2\phi}{\partial y^2} + d\frac{\partial\phi}{\partial x} + e\frac{\partial\phi}{\partial y} + f\phi + g = 0. \tag{1}$$

Equation (1) is classified into three categories by the values of a, b, and c as follows.

- $4ac > b^2$: Elliptic
  Example: Laplace equation
$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \tag{2}$$
- $4ac = b^2$: Parabolic
  Example: Diffusion equation
$$\frac{\partial\phi}{\partial t} = \frac{\partial^2\phi}{\partial x^2} \tag{3}$$
- $4ac < b^2$: Hyperbolic
  Example: Wave equation
$$\frac{\partial^2\phi}{\partial t^2} = \frac{\partial^2\phi}{\partial x^2} \tag{4}$$

Each of equations has different types of the solutions and the boundary conditions. Therefore, the numerical simulation techniques are individually proposed.

As the Helmholtz equation is classified as an elliptic partial differential equation, only the solution of elliptic PDE is described in the following sections.

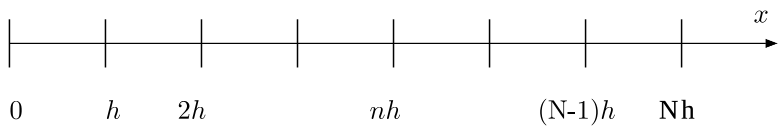

Figure 1: Discretization of 1D Helmholtz Equation

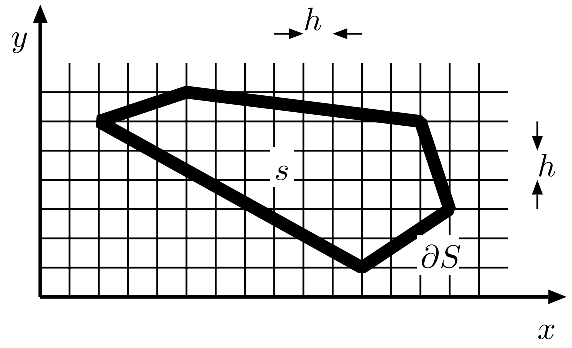

Figure 2: Finite Difference Grid of 2D Helmholtz Equation

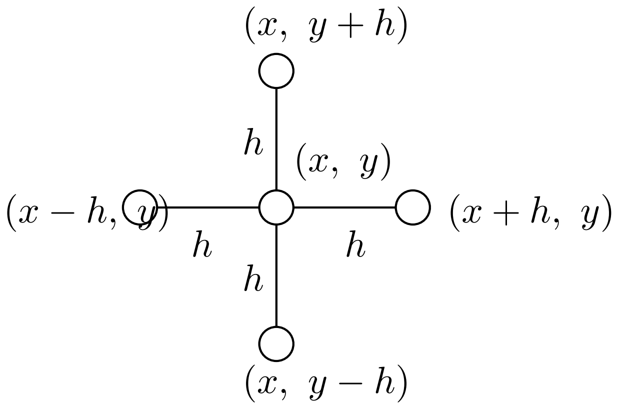

Figure 3: Finite Difference Approximation of 2D Helmholtz Equation

2 Finite Difference Method

2.1 Ordinary Differential Equation

For simplicity, the 1D Helmholtz equation with the Dirichlet boundary condition is considered as a first example.

$$\begin{aligned} \frac{\mathrm{d}^2}{\mathrm{d}x^2}\phi(x) + k^2\phi(x) &= -\rho(x) \quad (0 < x < a) \qquad (5) \\ \phi(0) &= \phi(a) = 0 \qquad (6) \end{aligned}$$

As shown in Fig. 1, the domain is evenly divided into N segments. The size of the segment is $h = \dfrac{a}{N}$. Considering the Taylor expansion

$$\phi(x \pm h) = \phi(x) \pm h\frac{\mathrm{d}}{\mathrm{d}x}\phi(x) + \frac{1}{2}h^2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\phi(x) \pm \frac{1}{6}h^3\frac{\mathrm{d}^3}{\mathrm{d}x^3}\phi(x) + \cdots, \qquad (7)$$

first and second derivatives are approximated by the following finite differences.

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\phi(x) &= \frac{1}{2h}\left\{\phi(x+h) - \phi(x-h)\right\} + \mathcal{O}(h^{-2}), \qquad (8) \\ \frac{\mathrm{d}^2}{\mathrm{d}x^2}\phi(x) &= \frac{1}{h^2}\left\{\phi(x+h) - 2\phi(x) + \phi(x-h)\right\} + \mathcal{O}(h^{-2}), \qquad (9) \end{aligned}$$

where $\mathcal{O}$ is the order of the residual. It is noted that both Eqs. (8) and (9) are said to be in the second order accuracy.

By neglecting $\mathcal{O}(h^{-2})$ as the infinitesimally small term, Eq. (5) is approximated at the grid point $x_n = nh$ in Fig. 1 as

$$\begin{aligned} \phi(x_n + h) + \phi(x_n - h) - \left\{2 - (kh)^2\right\}\phi(x_n) &= -h^2\rho(x_n), \qquad (10) \\ n &= 1, 2, \ldots, N-1. \end{aligned}$$

By substituting the boundary condition (6) into Eq. (11), $(N-1)$ linear equations with $(N-1)$ unknowns are obtained, and the unique solution exists.

2.2 Partial Differential Equation

Next, 2D Helmholtz equation with the Dirichlet boundary condition is considered.

$$\begin{aligned} \frac{\partial^2}{\partial x^2}\phi(x,\ y) + \frac{\partial^2}{\partial y^2}\phi(x,\ y) + k^2\phi(x,\ y) &= -\rho(x,\ y) \quad ((x,\ y) \in S), \qquad (11) \\ \phi(x_b,\ y_b) &= 0 \quad ((x_b,\ y_b) \in \partial S), \qquad (12) \end{aligned}$$

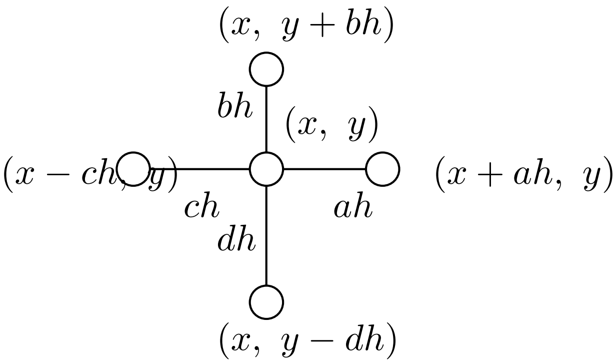

Figure 4: Finite Difference Approximation for Unevenly-Spaced Grid

The same as 1D case, the domain is discretized. As shown in Fig. 2, the domain is discretized into the grids with the spacing h along x- and y-axes. For the grid point of Fig. 3, Taylor expansion is applied as

$$\phi(x \pm h,\ y) = \phi(x,\ y) \pm h\frac{\partial}{\partial x}\phi(x,\ y) + \frac{1}{2}h^2\frac{\partial^2}{\partial x^2}\phi(x,\ y) \pm \frac{1}{6}h^3\frac{\partial^3}{\partial x^3}\phi(x,\ y) + \cdots, \tag{13}$$

$$\phi(x,\ y \pm h) = \phi(x,\ y) \pm h\frac{\partial}{\partial y}\phi(x,\ y) + \frac{1}{2}h^2\frac{\partial^2}{\partial y^2}\phi(x,\ y) \pm \frac{1}{6}h^3\frac{\partial^3}{\partial y^3}\phi(x,\ y) + \cdots. \tag{14}$$

Then, second derivative is approximated by the following finite differences.

$$\frac{\partial^2}{\partial x^2}\phi(x,\ y) = \frac{1}{h^2}\left\{\phi(x+h,\ y) - 2\phi(x,\ y) + \phi(x-h,\ y)\right\} + \mathcal{O}(h^{-2}), \tag{15}$$

$$\frac{\partial^2}{\partial y^2}\phi(x,\ y) = \frac{1}{h^2}\left\{\phi(x,\ y+h) - 2\phi(x,\ y) + \phi(x,\ y-h)\right\} + \mathcal{O}(h^{-2}). \tag{16}$$

Both Eqs. (15) and (16) are in the second order accuracy.

By neglecting $\mathcal{O}(h^{-2})$ as the infinitesimally small term, Eq. (11) is approximated at the grid point $(x_m,\ y_n)$ in Fig. 2 as

$$\phi(x_m+h,\ y_n)+\phi(x_m-h,\ y_n)+\phi(x_m,\ y_n+h)+\phi(x_m,\ y_n-h)-\left\{4-(kh)^2\right\}\phi(x_m,\ y_n) = -h^2\rho(x_m,\ y_n). \tag{17}$$

As is 1D case, a set of linear equations with the same number of unknowns is obtained by combining Eq. (17) with the Dirichlet boundary condition (12).

2.3 Unevenly-Spaced Grid

When the evenly-spaced grid is used, the boundary may not necessarily be on the grid points. In the case, the matching of the nodes with the boundary by using the unevenly-spaced grid is necessary.

For the grid points of Fig. 4, Taylor expansion is applied at $(x+ah,\ y)$ and $(x-ch,\ y)$ as

$$\phi(x+ah,\ y) = \phi(x,\ y) + (ah)\frac{\partial}{\partial x}\phi(x,\ y) + \frac{1}{2}(ah)^2\frac{\partial^2}{\partial x^2}\phi(x,\ y) + \frac{1}{6}(ah)^3\frac{\partial^3}{\partial x^3}\phi(x,\ y) + \cdots, \tag{18}$$

$$\phi(x-ch, y) = \phi(x,\ y) - (ch)\frac{\partial}{\partial x}\phi(x,\ y) + \frac{1}{2}(ch)^2\frac{\partial^2}{\partial x^2}\phi(x,\ y) - \frac{1}{6}(ch)^3\frac{\partial^3}{\partial x^3}\phi(x,\ y) + \cdots. \tag{19}$$

Next, by canceling the term of $\partial\phi/\partial x$ from Eq. (18) and (19), the following relation is obtained.

$$\frac{\partial^2}{\partial x^2}\phi(x,\ y) = \frac{2}{ac(a+c)h^2}\left\{c\phi(x+ah,\ y) + a\phi(x-ch,\ y) - (a+c)\phi(x,\ y)\right\} + \frac{(c-a)h}{3}\frac{\partial^3}{\partial x^3}\phi(x,\ y) + \mathcal{O}(h^{-2}). \tag{20}$$

By neglecting the third and higher derivative terms, the accuracy is second order when $a = c$, and is first order in other case. Therefore, the introduction of the non-evenly-spaced grid degrades the accuracy of the finite difference approximation.

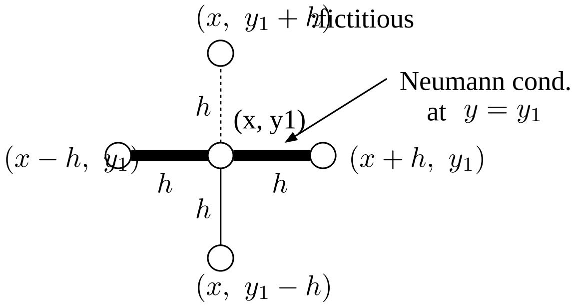

Figure 5: Finite Difference Approximation of Neumann Condition

In the same manner, the second derivative with respect to y is given as

$$\frac{\partial^2}{\partial y^2}\phi(x,\ y) = \frac{2}{bd(b+d)h^2}\left\{d\phi(x,\ y+bh) + b\phi(x,\ y-dh) - (b+d)\phi(x,\ y)\right\} + \frac{(d-b)h}{3}\frac{\partial^3}{\partial y^3}\phi(x,\ y) + \mathcal{O}(h^{-2}). \tag{21}$$

The accuracy is second order when $b = d$, and is first order in other case.

By substituting Eqs. (20) and (21) into Helmholtz equation, the finite difference approximation at $(x_m,\ y_n)$ is obtained.

$$2\left\{\frac{\phi(x_m+ah,\ y_n)}{a(a+c)} + \frac{\phi(x,\ y+bh)}{b(b+d)} + \frac{\phi(x-ch,\ y)}{c(a+c)} + \frac{\phi(x,\ y-dh)}{d(b+d)}\right\} - \left\{\frac{2(ac+bd)}{abcd} - (kh)^2\right\}\phi(x,\ y) \simeq -h^2\rho(x,\ y). \tag{22}$$

Equation (22) becomes identical to Eq. (17) when $a = b = c = d = 1$.

2.4 Neumann Condition

When the boundary satisfies the Dirichlet condition, the value of the wave function is given on the grid point. However, the care shall be taken if the boundary satisfies the Neumann condition since only the normal derivative of the boundary is given.

Considering Fig. 5, the domain is $y \le y_1$ and the Neumann condition is satisfied at $y = y_1$ as

$$\left.\frac{\partial}{\partial y}\phi(x,\ y)\right|_{y=y_1} = 0. \tag{23}$$

By using the virtual node at $(x,\ y_1 + h)$, the finite difference approximation of Helmholtz equation at $(x,\ y_1)$ is expressed as

$$\phi(x+h,\ y_1) + \phi(x-h,\ y_1) + \phi(x,\ y_1+h) + \phi(x,\ y_1-h) - \left\{4 - (kh)^2\right\}\phi(x,\ y) = -h^2\rho(x,\ y_1). \tag{24}$$

The Neumann condition (23) at $(x,\ y_1)$ is also expressed by using the virtual node $(x,\ y_1 + h)$ as

$$\begin{aligned} \left.\frac{\partial}{\partial y}\phi(x,\ y)\right|_{y=y_1} &\simeq \frac{1}{2h}\left\{\phi(x,\ y_1+h) - \phi(x,\ y_1-h)\right\} \\ &= 0. \end{aligned} \tag{25}$$

Therefore, $\phi(x,\ y_1 + h)$ can be eliminated from Eqs. (24) and (25). The resultant finite difference approximation at $(x_m,\ y_1)$ is expressed as

$$\phi(x_m+h,\ y_1) + \phi(x_m-h,\ y_1) + 2\phi(x_m,\ y_1-h) - \left\{4 - (kh)^2\right\}\phi(x_m,\ y_1) = -h^2\rho(x_m,\ y_1). \tag{26}$$

Report

Do not forget to fill out the student ID, your department and lab names, as well as your name.

The handouts as well as the copies of the slides can be downloaded from the web.

`http://mobile.ss.titech.ac.jp/~takada/waves/`

Exercises

1. Derive the finite difference approximation of 3D Helmholtz equation.
2. Compare FEM and FDM by pointing out advantages and disadvantages.

References

[1] E. Kreiszig, Advanced Engineering Mathematics, 8th Eds., John Wiley & Sons, New York (1999).