Wave Theory II
(7) Physical Optics Approximation

Jun-ichi Takada (takada@ide.titech.ac.jp)

In this lecture, the physical optics approximation (PO), which is classified as a semi-analytical technique, is described. In the physical optics approximation, the equivalent sources on the scatterer are approximated by those for the infinitely large plane boundary, which is a canonical problem. The scattered field is then computed by substituting the approximate solution into Kirchoff-Huygens Principle. PO is often used to compute the back scatter.

1 Review of Kirchoff-Huygens Principle

The position vectors of the source and the observer are expressed as $\boldsymbol{rho}'$ and $\boldsymbol{\rho}$, respectively. The Green function $G(\boldsymbol{\rho},\ \boldsymbol{\rho}')$ for the scalar Helmholtz equation is defined in the following equation.

$$\nabla^2 G(\boldsymbol{\rho},\ \boldsymbol{\rho}') + k^2 G(\boldsymbol{\rho},\ \boldsymbol{\rho}') = -\delta(\boldsymbol{\rho} - \boldsymbol{\rho}'). \tag{1}$$

By using $G(\boldsymbol{\rho},\ \boldsymbol{\rho}')$, the solution $\phi(\boldsymbol{\rho})$ for the general inhomogeneous Helmholtz equation

$$\nabla^2 \phi(\boldsymbol{\rho}) + k^2 \phi(\boldsymbol{\rho}) = -\rho(\boldsymbol{\rho}) \tag{2}$$

is expressed by using the Green function as

$$\phi(\boldsymbol{\rho}) = \int_S G(\boldsymbol{\rho},\ \boldsymbol{\rho}')\rho(\boldsymbol{\rho}')\mathrm{d}S' + \oint_{\partial S} \left\{ G(\boldsymbol{\rho},\ \boldsymbol{\rho}')\nabla'\phi(\boldsymbol{\rho}') - \nabla' G(\boldsymbol{\rho},\ \boldsymbol{\rho}')\phi(\boldsymbol{\rho}') \right\} \mathrm{d}\mathbf{n}', \tag{3}$$

where S the domain under consideration, ∂S the boundary of S composed of a set of closed contours, and $\mathrm{d}\mathbf{n}$ is the outward unit normal vector of ∂S. The solution of the Helmholtz equation is uniquely determined when *(i) the source distribution* and *(ii) the wave function on the boundary.* The wave function on the boundary is, however, usually unknown. Therefore, it is the boundary element method that solves the boundary integral equation with respect to this unknown wave function.

Contrary in the physical optics, the wave function on the boundary is approximated by that for the infinite plane boundary which is a canonical solution.

2 Infinite Plane Boundary

The behavior of the wave function on the infinite plane boundary is first described as a canonical problem.

As shown in Fig. 1, infinite plane boundary is considered at $y = 0$.

The 2D semi-infinite space problem with a unit point source at $(0,\ y')$ in the domain $y \geq 0$ is considered.

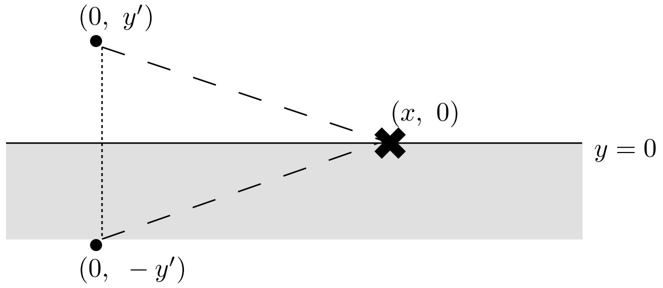

Figure 1: Treatment of infinite plane boundary

2.1 Free Space

In the free space, the wave function $\phi(\boldsymbol{\rho})$ generated from the unit point source at $(0,\ y')$ is expressed as

$$\begin{aligned}\phi(\boldsymbol{\rho}) &= G(\boldsymbol{\rho},\ \boldsymbol{\rho}') \\ &= \frac{1}{4j} H_0^{(2)}(k\rho_1), \qquad (4)\\ \rho_1 &= \sqrt{x^2+(y-y')^2}. \qquad (5)\end{aligned}$$

The derivative of the wave function along y, i.e. $\frac{\partial}{\partial y}\phi(\boldsymbol{\rho})$ is expressed as

$$\begin{aligned}\frac{\partial}{\partial y}\phi(\boldsymbol{\rho}) &= \frac{jk}{4} H_1^{(2)}(k\rho_1)\frac{\partial \rho_1}{\partial y} \\ &= \frac{jk}{4} H_1^{(2)}(k\rho_1)\frac{y-y'}{\rho_1}. \qquad (6)\end{aligned}$$

It is noted that

$$\frac{\mathrm{d}}{\mathrm{d}x} H_0^{(2)}(x) = -H_1^{(2)}(x) \qquad (7)$$

is used.

2.2 Semi-infinite Space

Next, the infinite plane boundary is considered. To satisfy the boundary condition, the image method is applied. The boundary is removed first, and then an image source at $(0,\ -y')$ is considered so as to satisfy the boundary condition at $y=0$. In the following discussion, the wave function due to the real source is denoted by $\phi_{\mathrm{d}}(\boldsymbol{\rho}),$ and that due to the image source is denoted by $\phi_{\mathrm{r}}(\boldsymbol{\rho})$.

2.2.1 Dirichlet Condition

To satisfy the Dirichlet condition

$$\phi(\boldsymbol{\rho}) = 0, \quad y=0, \qquad (8)$$

the amplitude of the image source shall be -1. In case,

$$\begin{aligned}\phi(\boldsymbol{\rho}) &= \phi_{\mathrm{d}}(\boldsymbol{\rho}) + \phi_{\mathrm{r}}(\boldsymbol{\rho}) \\ &= \frac{1}{4j} H_0^{(2)}(k\rho_1) - \frac{1}{4j} H_0^{(2)}(k\rho_2), \qquad (9)\end{aligned}$$

where

$$\rho_2 = \sqrt{x^2+(y+y')^2}. \qquad (10)$$

It is clear that $\rho_1 = \rho_2 \overset{\text{def}}{=} \rho_0$ at $y=0$, then $\phi(\boldsymbol{\rho})|_{y=0} = 0$ is obtained. In case, the normal derivative $\frac{\partial}{\partial y}\phi(\boldsymbol{\rho})$ is expressed as

$$\begin{aligned}\left.\frac{\partial}{\partial y}\phi(\boldsymbol{\rho})\right|_{y=0} &= \left.\frac{\partial}{\partial y}\phi_{\mathrm{d}}(\boldsymbol{\rho})\right|_{y=0} + \left.\frac{\partial}{\partial y}\phi_{\mathrm{r}}(\boldsymbol{\rho})\right|_{y=0} \\ &= \frac{jk}{4} H_1^{(2)}(k\rho_1)\frac{-y'}{\rho_0} - \frac{jk}{4} H_1^{(2)}(k\rho_2)\frac{y'}{\rho_0} \\ &= 2\left.\frac{\partial}{\partial y}\phi_{\mathrm{d}}(\boldsymbol{\rho})\right|_{y=0}, \qquad (11)\end{aligned}$$

where

$$\rho_0 = \sqrt{x^2+y'^2}. \qquad (12)$$

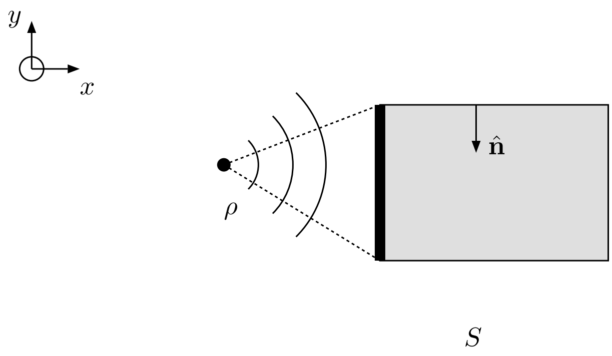

Figure 2: Physical Optics Approximation Model
An equivalent source is assumed only along the bold line.

2.2.2 Neumann Condition

To satisfy the Neumann condition

$$\frac{\partial}{\partial y}\phi(\boldsymbol{\rho}) = 0, \quad y = 0, \tag{13}$$

the amplitude of the image source shall be +1. In case,

$$\begin{aligned}\frac{\partial}{\partial y}\phi(\boldsymbol{\rho}) &= \frac{\partial}{\partial y}\phi_{\mathrm{d}}(\boldsymbol{\rho}) + \frac{\partial}{\partial y}\phi_{\mathrm{r}}(\boldsymbol{\rho}) \\ \frac{jk}{4}H_1^{(2)}(k\rho_1)\frac{-y'}{\rho_0} &+ \frac{jk}{4}H_1^{(2)}(k\rho_2)\frac{y'}{\rho_0}.\end{aligned} \tag{14}$$

It is clear that $\rho_1 = \rho_2 \stackrel{\mathrm{def}}{=} \rho_0$ at $y = 0$, then $\frac{\partial}{\partial y}\phi(\boldsymbol{\rho})\Big|_{y=0} = 0$ is obtained. In case, the wave function $\phi(\boldsymbol{\rho})$ is expressed as

$$\begin{aligned}\phi(\boldsymbol{\rho})|_{y=0} &= \phi_{\mathrm{d}}(\boldsymbol{\rho})|_{y=0} + \phi_{\mathrm{r}}(\boldsymbol{\rho})|_{y=0} \\ &= \frac{1}{4j}H_0^{(2)}(k\rho_0) + \frac{1}{4j}H_0^{(2)}(k\rho_0) \\ &= 2\phi_{\mathrm{d}}(\boldsymbol{\rho})|_{y=0}.\end{aligned} \tag{15}$$

3 Physical Optics Approximation

Although the equivalent source in the contour integral of Eq. (3) is unknown, the same relation as is described in the previous section is applied to approximate the equivalent source by using the incident wave as

Dirichlet condition

$$\phi(\boldsymbol{\rho}') \rightarrow 0, \tag{16}$$

$$\frac{\partial\phi(\boldsymbol{\rho}')}{\partial n'} \rightarrow \begin{cases} 2\frac{\partial\phi^{\mathrm{inc}}(\boldsymbol{\rho}')}{\partial n'}, & \text{visible point from the source,} \\ 0, & \text{otherwise.}\end{cases} \tag{17}$$

Neumann condition

$$\phi(\boldsymbol{\rho}') \rightarrow \begin{cases} 2\phi^{\mathrm{inc}}(\boldsymbol{\rho}'), & \text{visible point from the source,} \\ 0, & \text{otherwise,}\end{cases} \tag{18}$$

$$\frac{\partial\phi(\boldsymbol{\rho}')}{\partial n'} \rightarrow 0. \tag{19}$$

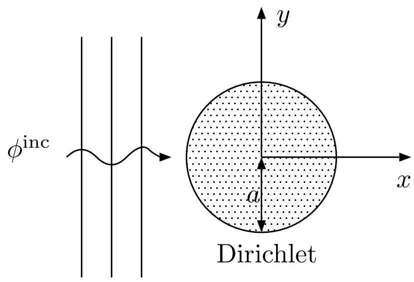

Figure 3: Plane wave scattering from infinite circular cylinder

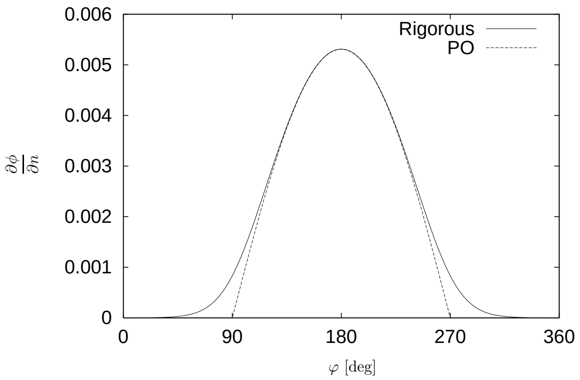

Figure 4: Equivalent secondary source on the scatterer (amplitude)
$(a = 5\lambda, d = 100\lambda)$

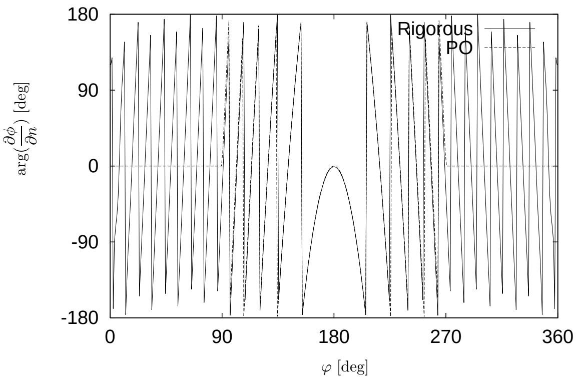

Figure 5: Equivalent secondary source on the scatterer (phase)
$(a = 5\lambda, d = 100\lambda)$

This approximation is called the *physical optics approximation (PO)*.

As PO uses the geometrical optics incident wave to compute the scattered wave, it is classified as a high frequency technique. High frequency techniques are advantageous when the scatterer is large relative to the wavelength.

4 Properties of PO

- Different from geometrical optics approximation, the wave function is continuous at the reflection and the shadow boundaries. The same computation is applicable independent of the position of the observer.
- It is time consuming to integrate the equivalent source which exists on the visible region of the boundary. However, the contribution from reflection and diffraction points can be extracted by using the stationary phase method.
- The error is larger outside the specular reflection region, especially in the shadow region. This is due to the inaccuracy of the equivalent source at the diffracting edges.

5 Numerical Example

As a 2D example, the scattering wave from the infinitely long circular cylinder with the Dirichlet condition for the $+x$-propagating plane wave incidence is considered. The analysis model is shown in Fig. 3. Since the problem of Fig. 3 can be solved analytically by using eigenfunction expansion method, the accuracy of the PO approximation can be evaluated by comparison.

Figures 4 and 5 compares respectively the amplitude and the phase of the rigorous and PO equivalent secondary sources on the boundary. The rigorous solution has nonzero amplitude in the shadowed region, but the PO approximation is zero in the region.

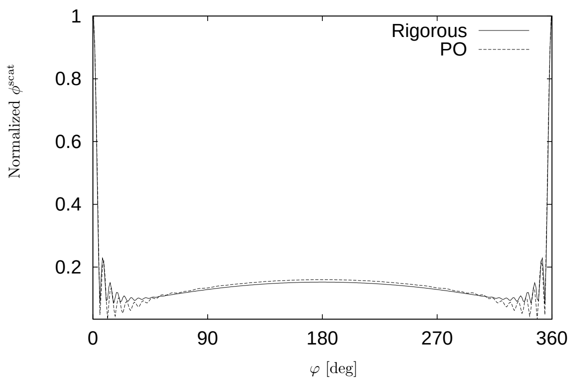

Figure 6: Directive pattern of the scattered wave in the far field region
$(a = 5\lambda, d = 100\lambda)$

Figure 6 shows the far field directive pattern of the scattered wave which is normalized by the maximal. In the back scattering region $\phi = 90^\circ \sim 270^\circ$ and the GO boundaries $\phi = 0^\circ, 360^\circ$, PO approximation gives the accurate results.

Report

Do not forget to fill out the student ID, your department and lab names, as well as your name.

The handouts as well as the copies of the slides can be downloaded from the web.

`http://mobile.ss.titech.ac.jp/~takada/waves/`

Exercises

1. Compute the scattering wave from the infinitely long circular cylinder with the Neumann condition for the $+x$-propagating plane wave incidence by using PO. If possible, compare with the analytical result as well.

References

[1] M. Ando, "Physical Optics," in E. Yamashita (Eds.), **Analysis Methods for Electromagnetic Wave Problems, Volume Two**, Chap. 4, Artech-House, 1995.